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OPTIMAL CONTROL FOR A CLASS OF PARTIALLY OBSERVED BILINEAR STOCHASTIC SYSTEMS

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ABSTRACT

An alternative formulation is presented for a class of partially observed bilinear stochastic control problem which is described by three sets of stochastic differential equations: one for the system to be controlled, one for the observer, and one for the control process which is driven by the observation process. With this formulation, the stochastic control problem is converted to an equivalent deterministic identification problem of control gain matrices. Using standard variation arguments, we obtain the necessary conditions of optimality on the basis of which the optimal control parameters can be determined.

I. INTRODUCTION

The Hamilton-Jacobi-Bellman (HJB) equation arising from the application of Bellman's principle of optimality as well as Ito's formula to controlled stochastic systems has been the major tool for determining optimal control laws (cf. [3,8]). With this approach, one is required to solve a nonlinear partial differential eaquation (of parabolic type) on the state space R^n. This has, so far, posed a major stumbling block in its application to engineering problems. It seems almost impossible to avoid solving the HJB-equation if one is interested in determining optimal controls for nonlinear, bilinear or even linear problems with control constraints.

Recently, Teo, Ahmed and Fisher [1] and Ahmed [2] proposed a new formulation for stochastic control problem for partially observed linear systems governed by stochastic differential equations driven by point processes [1] or general martingales [2]. With this formulation the stochastic control problem can be converted into an equivalent (deterministic) identification problem the necessary conditions of which can be obtained by use of standard variation arguments.

In this paper we consider the optimal control problem for a class of partially observed bilinear stochastic systems driven by standard Wiener processes. Using similar control structure as that of [1] or [2], we convert the original stochastic control problem into an equivalent identification problem for control gain matrices. Further, using standard varitional arguments (cf. [4,5,6]), we derive the necessary conditions of optimality on the basis of which the required optimal control parameters can be determined.

II. PROBLEM STATEMENT

Consider the following (bilinear) stochastic system

$$dx(t) = A(t)x(t)dt + B(t)du(t) + \sum_{i=1}^{M} \sigma_i(t)x(t)dW_i(t), \qquad x(0) = x_0, \quad t \in I \equiv [0,T], \tag{1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times r}$, $\sigma_i \in R^{n\times n}$; $1 \le i \le M$, and $W \equiv \{W_i; 1 \le i \le M\}$ is standard Wiener process with values in R^M. The initial state x_0 is a random variable independent of W. The control process $u(t); t \ge 0$ will be defined shortly. Along with the system (1), let the observed process be governed by

$$dy(t) = H_1(t)x(t)dt + H_2(t)y(t)dt + \sum_{i=1}^{N} \widetilde{\sigma}_i(t)y(t)dV_i(t), \quad t \in I, \qquad y(0) = 0, \tag{2}$$

where $H_1 \in R^{m\times n}$, $H_2 \in R^{m\times m}$, $\widetilde{\sigma}_i \in R^{m\times m}$ and $V \equiv \{V_i; 1 \le i \le N\}$ is an R^N-valued standard Wiener process independent of x_0 and W. Assuming that all the random vectors and processes described above are defined on some complete probability space $(\Omega, \mathcal{B}, \mu)$, we wish to design a control system having the structure

$$du(t) = K_1(t)y(t)dt + K_2(t)dy(t), \quad t \in I, \qquad u(0) = 0, \tag{3}$$

where the control parameters $K \equiv \{K_1, K_2\}$ are chosen from some admissible class $\mathcal{P} \subset L_\infty(I, R^{r\times m}) \times L_\infty(I, R^{r\times m})$ so that a given objective functional is minimized. As indicated in [2], one may consider several possible objective functionals related to the observed process y.

(OF1) : $J_1(K) \equiv E \int_I \Big(Q(t)y(t), y(t)\Big) dt$

(OF2) : $J_2(K) \equiv E \int_I \Big(Q(t)(y(t) - y_d(t)), y(t) - y_d(t)\Big) dt$

(OF3) : $J_3(K) \equiv \int_I \Big(Q(t)(\widehat{y}(t) - y_d(t)), \widehat{y}(t) - y_d(t)\Big) dt$

(OF4) : $J_4(K) \equiv E \int_I \Big(Q(t)(y(t) - \widehat{y}(t)), y(t) - \widehat{y}(t)\Big) dt$

where $\widehat{y}(t) \equiv E\{y(t)\}$, $(\cdot,\cdot)$ denotes the scalar product in R^m, and Q is any continuous positive semidefinite symmetric matrix-valued function and y_d is the desired output. In this paper we shall only deal with the objective functional J_1. The rest can be dealt with in a similar manner.

Assumptions

(A1) all the matrix-valued functions A, B, σ_i, H_1, H_2, and $\widetilde{\sigma}_i$ are continuous.

(A2) the control parameter $K \equiv \{K_1, K_2\} \in \mathcal{P}$ where $\mathcal{P}$ is a compact convex subset of $L_\infty(I, R^{r\times m}) \times L_\infty(I, R^{r\times m})$.

III. FORMULATION OF IDENTIFICATION PROBLEM

Consider the following control problem

(SP1) Find a $K \in \mathcal{P}$ such that

$$J_1(K) \equiv E \int_I \Big(Q(t)y(t), y(t)\Big) dt \equiv \min. \tag{4}$$

subject to the constraints (1)-(3).

Here Q is a continuous positive semidefinite symmetric matrix valued function. In this section we shall show how the problem (SP1) can be converted into an equivalent (deterministic) identification problem. First using (3) in (1), we obtain

$$\begin{aligned} dx(t) &= (A(t) + B(t)K_2(t)H_1(t))x(t)dt \\ &\quad + (B(t)K_1(t) + B(t)K_2(t)H_2(t))y(t)dt \\ &\quad + \sum_{i=1}^{M} \sigma_i(t)x(t)dW_i(t) + \sum_{i=1}^{N} B_2(t)K_2(t)\widetilde{\sigma}_i(t)y(t)dV_i(t), \\ x(0) &= x_0, \quad t \in I. \end{aligned} \tag{5}$$

Defining $\xi \equiv (x,y)'$, the system (5) together with the observation dynamics (2) can jointly be written as

$$d\xi(t) = \mathcal{A}(t,K)\xi(t)dt + \sum_{i=1}^{M} C_i(t)\xi(t)dW_i(t) + \sum_{i=1}^{N} D_i(t,K)\xi(t)dV_i(t), \qquad \xi(0) = \xi_0, \quad t \in I, \tag{6}$$

where

$$\mathcal{A}(t,K) = \begin{pmatrix} A(t) + B(t)K_2(t)H_1(t) & B(t)[K_1(t) + K_2(t)H_2(t)] \\ H_1(t) & H_2(t) \end{pmatrix} \tag{7}$$

$$C_i(t) = \begin{pmatrix} \sigma_i(t) & 0 \\ 0 & 0 \end{pmatrix}, \qquad D_i(t,K) = \begin{pmatrix} 0 & B(t)K_2(t)\widetilde{\sigma}_i(t) \\ 0 & \widetilde{\sigma}_i(t) \end{pmatrix}. \tag{8}$$

For the solution of the problem (SP1), we shall need the following result.

Lemma 1

Consider the system (6) and suppose ξ_0, W and V are statistically independent. Then for each $K \in \mathcal{P}$, the mean $\widehat{\xi}(t) \equiv E\{\xi(t)\}$ and the covariance $P(t) \equiv E\{(\xi(t) - \widehat{\xi}(t))(\xi(t) - \widehat{\xi}(t))'\}$, $t \ge 0$, satisfy the following system of differential equations

$$\frac{d}{dt}\widehat{\xi}(t) = \mathcal{A}(t,K)\widehat{\xi}(t), \quad t \in I, \qquad \widehat{\xi}(0) = \widehat{\xi}_0 \equiv (\widehat{x}_0, 0)', \tag{9}$$

and

$$\dot{P}(t) = \mathcal{A}(t,K)P(t) + P(t)\mathcal{A}'(t,K) + \sum_{i=1}^{M} C_i(t)[P(t) + \hat{\xi}(t)\hat{\xi}'(t)]C_i'(t)$$
$$+ \sum_{i=1}^{N} D_i(t,K)[P(t) + \hat{\xi}(t)\hat{\xi}'(t)]D_i'(t,K), \quad t \in I,$$
$$P(0) = P_0. \tag{10}$$

Proof

The proof follows from standard computations. ■

Remark 1

For each $K \in \mathcal{P}$, the covariance matrix $P(t) \equiv P(t,K)$, $t \geq 0$, can be partitioned as

$$P(t) = \begin{pmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{pmatrix},$$

where

$$P_{11}(t) \equiv E\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))'\},$$
$$P_{22}(t) \equiv E\{(y(t) - \hat{y}(t))(y(t) - \hat{y}(t))'\},$$
$$P_{12}(t) = P_{21}(t) \equiv E\{(x(t) - \hat{x}(t))(y(t) - \hat{y}(t))'\}.$$

Using the result of Lemma 1, we can convert the problem (SP1) into equivalent (deterministic) identification problem. Let

$$\Lambda(t) \equiv P(t) + \hat{\xi}(t)\hat{\xi}'(t), \quad t \geq 0. \tag{11}$$

Then using (9) and (10), one can easily verify that Λ satisfies the following (matrix) differential equation

$$\frac{d}{dt}\Lambda(t) = \mathcal{A}(t,K)\Lambda(t) + \Lambda(t)\mathcal{A}'(t,K) + \sum_{i=1}^{M} C_i(t)\Lambda(t)C_i'(t)$$
$$+ \sum_{i=1}^{N} D_i(t,K)\Lambda(t)D_i'(t,K), \quad t \in I, \ \Lambda(0) = \Lambda_0. \tag{12}$$

Defining $\bar{Q} = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$, it follows that

$$J_1(K) = \int_I tr\Big(\bar{Q}(t)\Lambda(t)\Big)dt. \tag{13}$$

Using (13) it is clear that the problem (SP1) can be restated as follows

(DP1) Find a $K \in \mathcal{P}$ such that

$$J_1(K) \equiv \int_I tr\Big(\bar{Q}(t)\Lambda(t)\Big)dt = \min.$$

subject to the dynamic constraint (12).

We close this section by indicating that one can show, using similar arguments as those of Ahmed [2-6], that the problem (DP1) has a solution.

IV. NECESSARY CONDITIONS OF OPTIMALITY

In this section we utilize standard variation arguments to derive the necessary conditions of optimality for the problem (DP1). Let $K^o \in \mathcal{P}$ be the optimal parameter for the problem (DP1) and let $\Lambda^o(t) \equiv \Lambda(t,K^o), t \geq 0$, be the solution of (12) corresponding to K^o. Suppose the parameter set $\mathcal{P}$ is convex and let $\tilde{\Lambda}(t) \equiv \tilde{\Lambda}(t,K^o;K-K^o)$, $t \geq 0$, denote the Gateaux differential of Λ at K^o in the direction $(K - K^o)$. For the derivation of the necessary conditions of optimality we shall need the following result the proof of which follows from standard computations.

Lemma 2

Consider the problem (DP1) and suppose that the parameter set $\mathcal{P}$ is convex. Then for each pair $K, K^o \in \mathcal{P}$, the Gateaux differential of J_1, at K^0 in the direction $(K - K^o)$, exists and is given by

$$J_1'(K^o;K-K^o) \equiv \int_I tr\Big(\bar{Q}(t)\tilde{\Lambda}(t)\Big)dt \geq 0. \tag{14}$$

Here $\tilde{\Lambda}$ denotes the Gateaux differential of Λ satisfying

$$\frac{d}{dt}\tilde{\Lambda}(t) = \mathcal{A}(t,K^o)\tilde{\Lambda}(t) + \tilde{\Lambda}(t)\mathcal{A}'(t,K^o) + \sum_{i=1}^{M} C_i(t)\tilde{\Lambda}(t)C_i'(t)$$
$$+ \sum_{i=1}^{N} D_i(t,K^o)\tilde{\Lambda}(t)D_i'(t,K^o) + \tilde{\mathcal{A}}(t)\Lambda^o(t) + \Lambda^o(t)\tilde{\mathcal{A}}'(t)$$
$$+ \sum_{i=1}^{N}[D_i(t,K^o)\Lambda^o(t)\tilde{D}_i'(t) + \tilde{D}_i(t)\Lambda^o(t)D_i'(t,K^o)] \tag{15}$$
$$\tilde{\Lambda}(0) = 0, \quad t \in I,$$

where $\tilde{\mathcal{A}}$ and $\tilde{D}_i$ denote the Gateaux differential of $\mathcal{A}$ and D_i. ■

With the help of the above Lemma, we present the following necessary conditions of optimality.

Theorem 3

Consider the problem (DP1) and suppose Lemma 2 hold. Then the optimal parameter $K^o \in \mathcal{P}$ can be determined by the simultaneous solution of the differential equation

$$\frac{d}{dt}\Lambda^o(t) = \mathcal{A}(t,K^o)\Lambda^o(t) + \Lambda^o(t)\tilde{\mathcal{A}}'(t,K^o) + \sum_{i=1}^{M} C_i(t)\Lambda^o(t)C_i'(t)$$
$$+ \sum_{i=1}^{N} D_i(t,K^o)\Lambda^o(t)D_i'(t,K^o) \quad ,t \in I, \ \Lambda^o(0) = \Lambda_0, \tag{16}$$

the adjoint equation

$$-\frac{d}{dt}\Gamma^o(t) = \mathcal{A}'(t\ \ K^o)\Gamma^o(t) + \Gamma^o(t)\mathcal{A}(t,K^o) + \sum_{i=1}^{M} C_i'(t)\Gamma^o(t)C_i(t)$$
$$+ \sum_{i=1}^{N} D_i'(t,K^o)\Gamma^o(t)D_i(t,K^o) + \bar{Q}(t), \quad t \in I, \ \Gamma^o(T) = 0, \tag{17}$$

and the inequality

$$\int_I tr\Big(\Gamma^o(t)\tilde{\mathcal{A}}(t)\Lambda^o(t) + \sum_{i=1}^{N} \Gamma^o(t)D_i'(t,K^o)\Lambda^o(t)\tilde{D}_i(t)\Big)dt \geq 0. \tag{18}$$

Proof

In order that J_1, as defined by (13), attains its minimum at $K^o \in \mathcal{P}$, it is necessary that (14) holds for all $K \in \mathcal{P}$, where $\tilde{\Lambda}$ denotes the Gateaux differential of Λ as defined by Lemma 2. The inequality (14) can be further simplified by introducing the adjoint variable Γ^o which is the solution of the (matrix) differential equation (17). Using (14), (15) and (17), one can easily verify that

$$\int_I tr\Big(\bar{Q}(t)\tilde{\Lambda}(t)\Big)dt = 2\int_I tr\Big(\Gamma^o(t)\tilde{\mathcal{A}}(t)\Lambda^o(t)$$
$$+ \sum_{i=1}^{N} \Gamma^o(t)D_i'(t,K^o)\Lambda^o(t)\tilde{D}_i(t)\Big)dt. \tag{19}$$

Now the inequality (18) follows from (14) and (19). ■

Remark 2

Based on the above necessary conditions one can compute the optimal parameter K^o using any of the algorithms proposed in [3 or 7].

V. CONCLUSION

In this paper we have considered the optimal control problem for a class of partially observed bilinear stochastic systems. Assuming that the control process is governed by a (stochastic) differential equation, driven by the output process, with unknown parameters, we have converted the original stochastic control problem into an equivalent identification problem for control parameters. Using standard variational arguments, we derived the corresponding necessary conditions of optimality on the basis of which optimal parameters and hence optimal control can be determined.

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